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Persistent currents for the 2D Schrödinger operator with a strong δ-interaction on a loop

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Abstract

We investigate the two-dimensional magnetic Schrödinger operator $H_{B,\beta} = (-\mathrm{i}\nabla - A)^2 - \beta\delta(\cdot - \Gamma)$, where Γ is a smooth loop and the vector potential A corresponds to a homogeneous magnetic field B perpendicular to the plane. The asymptotics of negative eigenvalues of $H_{B,\beta}$ for $\beta \to \infty$ is found. It shows, in particular, that for large enough positive β the system exhibits persistent currents.

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1. Introduction

One of the most often studied features of mesoscopic systems is the persistent currents in rings threaded by a magnetic flux—see, e.g., [CGR, CWB] and scores of other theoretical and experimental papers where they were discussed. For a charged particle (an electron) confined to a loop Γ the effect is manifested by the dependence of the corresponding eigenvalues λ_n on the flux ϕ threading the loop, conventionally measured in the units of flux quanta, $2\pi\hbar c|e|^{-1}$. The derivative $\partial\lambda_n/\partial\phi$ equals $-\frac{1}{c}I_n$, where I_n is the persistent current in the nth state. In particular, if the particle motion on the loop is free, we have

$$\lambda_n(\phi) = \frac{\hbar^2}{2m^*}\left(\frac{2\pi}{L}\right)^2 (n+\phi)^2 \tag{1.1}$$

where L is the loop circumference and m^* is the effective mass of the electron, so the currents depend linearly on the applied field in this case.

Of course, the above example is idealized assuming that the particle is strictly confined to the loop. In reality, boundaries of a quantum wire are potential jumps at interfaces of different materials. As a consequence, electrons can be found outside the loop, even if not too far when we consider energies at which the exterior is a classically forbidden region.

A reasonable model respecting the essentially one-dimensional nature of quantum wires is a 2D Schrödinger operator with an attractive δ-interaction on an appropriate curve Γ, or more generally, a planar graph. Since the interaction support has codimension one, the Hamiltonian can be defined through its quadratic form and the corresponding resolvent can be written explicitly as a generalization of the Birman–Schwinger theory [BT, BEKŠ]. This leads to some interesting consequences such as the existence of bound states due to bending of an infinite and asymptotically straight curve [EI].

A natural question is how such a model is related to the ideal one in which the electron is strictly confined to the curve Γ. In [EY], we have derived an asymptotic formula showing that if the δ-coupling is strong, the negative eigenvalues approach those of the ideal model in which the geometry of Γ is taken into account by means of an effective curvature-induced potential. The purpose of this paper is to ask a similar question in the situation when the electron is subject to a homogeneous magnetic field B perpendicular to the plane. We are going to derive an analogous asymptotic formula where now the presence of the magnetic field is taken into account via the boundary conditions specifying the domain of the comparison operator.

An easy consequence of this result is that for a strong enough δ-interaction the negative eigenvalues of our Hamiltonian are not constant as functions of B, i.e. that the system exhibits persistent currents. Their further properties depend, of course, on the specific shape of Γ; this fact and the stability of such currents with respect to a disorder raise questions about optimal ways of interpreting the corresponding magnetic transport. We comment on this point in the concluding remarks.

2. Description of the model and the results

As we have explained above we are going to study the Schrödinger operator in $L^2(\mathbb{R}^2)$ with a constant magnetic field and an attractive δ-interaction on a loop. For the sake of simplicity, we employ rational units, $\hbar = c = 2m^* = 1$, and absorb the electron charge into the field intensity B. We shall use the circular gauge, $A(x, y) = \left(-\frac{1}{2}By, \frac{1}{2}Bx\right)$.

Let $\Gamma : [0, L] \ni s \mapsto (\Gamma_1(s), \Gamma_2(s)) \in \mathbb{R}^2$ be a closed counter-clockwise C^4 Jordan curve which is parametrized by its arc length. Given $\beta > 0$ and $B \in \mathbb{R}$, we define

$$q_{B,\beta}[f] = \left\|\left(-\mathrm{i}\partial_x + \tfrac{1}{2}By\right) f\right\|^2 + \left\|\left(-\mathrm{i}\partial_y - \tfrac{1}{2}Bx\right) f\right\|^2 - \beta \int_\Gamma |f(x)|^2 \,\mathrm{d}s$$

with the domain $H^1(\mathbb{R}^2)$, where $\partial_x \equiv \partial/\partial x$ etc, and the norm refers to $L^2(\mathbb{R}^2)$. It is straightforward to check that the form $q_{B,\beta}$ is closed and below bounded. We denote by $H_{B,\beta}$ the self-adjoint operator associated with it which can be formally written as

$$H_{B,\beta} = (-\mathrm{i}\nabla - A)^2 - \beta\delta(\cdot - \Gamma).$$

Our main aim is to study the asymptotic behaviour of the negative eigenvalues of $H_{B,\beta}$ as $\beta \to +\infty$.

Let $\gamma : [0, L] \ni s \mapsto (\Gamma_1''\Gamma_2' - \Gamma_2''\Gamma_1')(s) \in \mathbb{R}$ be the signed curvature of Γ. We denote by Ω the region enclosed by Γ, with the area $|\Omega|$, and define the operator

$$S_B = -\frac{\mathrm{d}^2}{\mathrm{d}s^2} - \frac{1}{4}\gamma(s)^2$$

on $L^2((0, L))$ with the domain

$$P_B = \left\{\varphi \in H^2((0, L));\ \varphi^{(k)}(L) = \exp(-\mathrm{i}B|\Omega|)\varphi^{(k)}(0),\ k = 0, 1\right\}$$

where $\varphi^{(k)}$ stands for the kth derivative.

We fix $j \in \mathbb{N}$ and denote by $\mu_j(B)$ the jth eigenvalue of S_B counted with multiplicity. Our main result is the following claim.

Theorem 2.1. *Let n be an arbitrary integer and let $\emptyset \neq I \subset \mathbb{R}$ be a compact interval. Then there exists $\beta(n, I) > 0$ such that*

$$\sharp\{\sigma_d(H_{B,\beta}) \cap (-\infty, 0)\} \geqslant n \qquad \text{for} \quad \beta \geqslant \beta(n, I) \quad \text{and} \quad B \in I.$$

For $\beta \geqslant \beta(n)$ and $B \in I$ we denote by $\lambda_n(B, \beta)$ the nth eigenvalue of $H_{B,\beta}$ counted with multiplicity. Then $\lambda_n(B, \beta)$ admits an asymptotic expansion of the form

$$\lambda_n(B, \beta) = -\tfrac{1}{4}\beta^2 + \mu_n(B) + \mathcal{O}(\beta^{-1} \ln \beta) \qquad \text{as} \quad \beta \to +\infty$$

where the error term is uniform with respect to $B \in I$.

Recall that the flux ϕ through the loop is $B|\Omega|/2\pi$ in our units. The existence of persistent currents is then given by the following consequence of the above result.

Corollary 2.2. *Let $\emptyset \neq I \subset \mathbb{R}$ be a compact interval and let $n \in \mathbb{N}$. Then there exists a constant $\beta_1(n, I) > 0$ such that the function $\lambda_n(\cdot, \beta)$ is not constant for $\beta \geqslant \beta_1(n, I)$.*

3. The proofs

Since the spectral properties of $H_{B,\beta}$ are clearly invariant with respect to Euclidean transformation of the plane, we may assume without any loss of generality that the curve Γ parametrizes in the following way:

$$\Gamma_1(s) = \Gamma_1(0) + \int_0^s \cos H(t)\, \mathrm{d}t \qquad\qquad \Gamma_2(s) = \Gamma_2(0) + \int_0^s \sin H(t)\, \mathrm{d}t$$

where $H(t) := -\int_0^t \gamma(u)\, \mathrm{d}u$. Let Ψ_a be the map

$$\Psi_a : [0, L) \times (-a, a) \ni (s, u) \mapsto (\Gamma_1(s) - u\Gamma_2'(s), \Gamma_2(s) + u\Gamma_1'(s)) \in \mathbb{R}^2.$$

From [EY, lemma 2.1] we know that there exists $a_1 > 0$ such that the map Ψ_a is injective for all $a \in (0, a_1]$. We thus fix $a \in (0, a_1)$ and denote by Σ_a the strip of width $2a$ enclosing Γ

$$\Sigma_a := \Psi_a\left([0, L) \times (-a, a)\right).$$

Then the set $\mathbb{R}^2 \setminus \Sigma_a$ consists of two connected components which we denote by Λ_a^{in} and Λ_a^{out}, where the interior one, Λ_a^{in}, is compact. We define a pair of quadratic forms,

$$q^{\pm}_{B,a,\beta}[f] = \left\|\left(-\mathrm{i}\partial_x + \tfrac{1}{2}By\right) f\right\|^2_{L^2(\Sigma_a)} + \left\|\left(-\mathrm{i}\partial_y - \tfrac{1}{2}Bx\right) f\right\|^2_{L^2(\Sigma_a)} - \beta \int_\Gamma |f(x)|^2\, \mathrm{d}s$$

which are given by the same expression but differ by their domains; the latter is $H_0^1(\Sigma_a)$ for $q^+_{B,a,\beta}$ and $H^1(\Sigma_a)$ for $q^-_{B,a,\beta}$. Furthermore, we introduce the quadratic forms

$$e^{j,\pm}_{B,a}[f] = \left\|\left(-\mathrm{i}\partial_x + \tfrac{1}{2}By\right) f\right\|^2_{L^2(\Lambda_a^j)} + \left\|\left(-\mathrm{i}\partial_y - \tfrac{1}{2}Bx\right) f\right\|^2_{L^2(\Lambda_a^j)}$$

for $j = \mathrm{out}, \mathrm{in}$, with the domains $H_0^1\left(\Lambda_a^j\right)$ and $H^1\left(\Lambda_a^j\right)$ corresponding to the $\pm$ sign, respectively. Let $L^{\pm}_{B,a,\beta}$, $E^{\mathrm{out},\pm}_{B,a}$ and $E^{\mathrm{in},\pm}_{B,a}$ be the self-adjoint operators associated with the forms $q^{\pm}_{B,a,\beta}$, $e^{\mathrm{out},\pm}_{B,a}$ and $e^{\mathrm{in},\pm}_{B,a}$, respectively.

As in [EY] we are going to use the Dirichlet–Neumann bracketing with additional boundary conditions at the boundary of Σ_a. It works in the magnetic case too as one can easily see comparing the form domains of the involved operators, cf [RS, thm XIII.2]. We get

$$E^{\mathrm{in},-}_{B,a} \oplus L^-_{B,a,\beta} \oplus E^{\mathrm{out},-}_{B,a} \leqslant H_{B,\beta} \leqslant E^{\mathrm{in},+}_{B,a} \oplus L^+_{B,a,\beta} \oplus E^{\mathrm{out},+}_{B,a} \tag{3.1}$$

with the decomposed estimating operators in $L^2(\mathbb{R}^2) = L^2\big(\Lambda_a^{\mathrm{in}}\big) \oplus L^2(\Sigma_a) \oplus L^2\big(\Lambda_a^{\mathrm{out}}\big)$. In order to assess the negative eigenvalues of $H_{B,\beta}$, it suffices to consider those of $L^+_{B,a,\beta}$ and $L^-_{B,a,\beta}$, because the other operators involved in (3.1) are positive. Since the loop is smooth, we can pass inside Σ_a to the natural curvilinear coordinates; we state

$$(U_a f)(s,u) = (1+u\gamma(s))^{1/2} f(\Psi_a(s,u)) \qquad \text{for} \quad f \in L^2(\Sigma_a)$$

which defines the unitary operator U_a from $L^2(\Sigma_a)$ to $L^2((0,L)\times(-a,a))$. To express the estimating operators in the new variables, we introduce

$$\begin{aligned} Q_a^+ &= \{\varphi \in H^1((0,L)\times(-a,a));\ \varphi(L,\cdot) = \varphi(0,\cdot) \text{ on } (-a,a),\\ &\qquad \varphi(\cdot,a) = \varphi(\cdot,-a) = 0 \text{ on } (0,L)\}\\ Q_a^- &= \{\varphi \in H^1((0,L)\times(-a,a));\ \varphi(L,\cdot) = \varphi(0,\cdot) \text{ on } (-a,a)\} \end{aligned}$$

and define the quadratic forms

$$\begin{aligned} b^{\pm}_{B,a,\beta}[g] &= \int_0^L\int_{-a}^a (1+u\gamma(s))^{-2}\,|\partial_s g|^2\ \mathrm{d}u\,\mathrm{d}s + \int_0^L\int_{-a}^a |\partial_u g|^2\ \mathrm{d}u\,\mathrm{d}s\\ &\quad + \int_0^L\int_{-a}^a V(s,u)|g|^2\,\mathrm{d}s\,\mathrm{d}u - \beta\int_0^L |g(s,0)|^2\,\mathrm{d}s\\ &\quad - \frac{b_\pm}{2}\int_0^L \frac{\gamma(s)}{1+a\gamma(s)}|g(s,a)|^2\,\mathrm{d}s + \frac{b_\pm}{2}\int_0^L \frac{\gamma(s)}{1-a\gamma(s)}|g(s,-a)|^2\,\mathrm{d}s\\ &\quad + \frac{1}{4}\int_0^L\int_{-a}^a B^2\left(\Gamma_1^2 - 2u\Gamma_1\Gamma_2' + \Gamma_2^2 + 2u\Gamma_2\Gamma_1' + u^2\right)|g|^2\,\mathrm{d}u\,\mathrm{d}s\\ &\quad + B\,\mathrm{Im}\int_0^L\int_{-a}^a (\Gamma_2 + u\Gamma_1')\left((1+u\gamma)^{-1}\cos H\bar{g}\partial_s g - \sin H\bar{g}\partial_u g\right)\mathrm{d}u\,\mathrm{d}s\\ &\quad - B\,\mathrm{Im}\int_0^L\int_{-a}^a (\Gamma_1 - u\Gamma_2')\left((1+u\gamma)^{-1}\sin H\bar{g}\partial_s g + \cos H\bar{g}\partial_u g\right)\ \mathrm{d}u\,\mathrm{d}s \end{aligned} \tag{3.2}$$

on $Q_a^\pm$, respectively, where $b_+ = 0$ and $b_- = 1$, and

$$V(s,u) = \tfrac{1}{2}(1+u\gamma(s))^{-3}u\gamma''(s) - \tfrac{5}{4}(1+u\gamma(s))^{-4}u^2\gamma'(s)^2 - \tfrac{1}{4}(1+u\gamma(s))^{-2}\gamma(s)^2$$

is the well-known curvature-induced effective potential [EŠ]. Let $D^{\pm}_{B,a,\beta}$ be the self-adjoint operators associated with the forms $b^{\pm}_{B,a,\beta}$, respectively. In analogy with [EY, lemma 2.2], we get the following result.

Lemma 3.1. $U_a^* D^{\pm}_{B,a,\beta} U_a = L^{\pm}_{B,a,\beta}$.

The presence of the magnetic field gave rise to terms containing $\bar{g}\partial_s g$ and $\bar{g}\partial_u g$ in (3.2). In order to eliminate the corresponding coefficients modulo small errors, we employ another unitary operator. We define

$$T_B(s) = -\frac{1}{2}B\int_0^s (\Gamma_2(t)\Gamma_1'(t) - \Gamma_2'(t)\Gamma_1(t))\,\mathrm{d}t.$$

It follows from the Green theorem that $T_B(L) = B|\Omega|$. Then we define

$$(M_B h)(s,u) := \exp\left[\mathrm{i}T_B(s) + \frac{\mathrm{i}}{2}Bu\left(\Gamma_2(s)\sin H(s) + \Gamma_1(s)\cos H(s)\right)\right]h(s,u)$$

for any $h \in L^2((0, L) \times (-a, a))$; it is straightforward to check that M_B is a unitary operator on $L^2((0, L) \times (-a, a))$. We define

$$\begin{aligned}\tilde{Q}^+_{B,a} &= \{\varphi \in H^1((0, L) \times (-a, a)); \varphi(L, \cdot) = \mathrm{e}^{-\mathrm{i}B|\Omega|}\varphi(0, \cdot) \text{ on } (-a, a),\\ &\qquad \varphi(\cdot, a) = \varphi(\cdot, -a) = 0 \text{ on } (0, L)\}\\ \tilde{Q}^-_{B,a} &= \{\varphi \in H^1((0, L) \times (-a, a)); \varphi(L, \cdot) = \mathrm{e}^{-\mathrm{i}B|\Omega|}\varphi(0, \cdot) \text{ on } (-a, a)\}\end{aligned}$$

and another pair of quadratic forms

$$\begin{aligned}\tilde{b}^{\pm}_{B,a,\beta}[g] = \int_0^L \int_{-a}^a &\{(1 + u\gamma)^{-2}|\partial_s g|^2 + |\partial_u g|^2 + [B(\Gamma_2 + u\Gamma_1')(1 + u\gamma)^{-1} \cos H\\ &- B(\Gamma_1 - u\Gamma_2')(1 + u\gamma)^{-1} \sin H - B(1 + u\gamma)^{-2}(\Gamma_2 \cos H - \Gamma_1 \sin H)\\ &+ B(1 + u\gamma)^{-2}(\Gamma_2 \sin H + \Gamma_1 \cos H)'u] \operatorname{Im}(\bar{g}\partial_s g) + W_B(s, u)|g|^2\} \,\mathrm{d}u \,\mathrm{d}s\\ &- \beta \int_0^L |g(s, 0)|^2 \,\mathrm{d}s - \frac{b_\pm}{2} \int_0^L \frac{\gamma(s)}{1 + a\gamma(s)}|g(s, a)|^2 \,\mathrm{d}s\\ &+ \frac{b_\pm}{2} \int_0^L \frac{\gamma(s)}{1 - a\gamma(s)}|g(s, -a)|^2 \,\mathrm{d}s\end{aligned}$$

for $g \in \tilde{Q}^{\pm}_{B,a}$, respectively, where

$$\begin{aligned}W_B(s, u) = V(s, u) &+ \tfrac{1}{4}(1 + u\gamma)^{-2}B^2u^2((\Gamma_2 \sin H + \Gamma_1 \cos H)')^2\\ &+ \tfrac{1}{4}B^2 \left(\Gamma_1^2 - 2u\Gamma_1\Gamma_2' + \Gamma_2^2 + 2u\Gamma_2\Gamma_1' + u^2\right)\\ &+ B(\Gamma_2 + u\Gamma_1')(1 + u\gamma)^{-1}T_B'(s) \cos H - B(\Gamma_1 - u\Gamma_2')(1 + u\gamma)^{-1}T_B'(s) \sin H\\ &+ \tfrac{1}{4}(1 + u\gamma)^{-2}B^2(\Gamma_2 \cos H - \Gamma_1 \sin H)^2 + \tfrac{1}{4}B^2(\Gamma_2 \sin H + \Gamma_1 \cos H)^2\\ &+ [B(\Gamma_2 + u\Gamma_1')(1 + u\gamma)^{-1} \cos H - B(\Gamma_1 - u\Gamma_2')(1 + u\gamma)^{-1} \sin H\\ &- B(1 + u\gamma)^{-2}(\Gamma_2 \cos H - \Gamma_1 \sin H)]\tfrac{1}{2}B(\Gamma_2 \sin H + \Gamma_1 \cos H)'u\\ &+ [-B(\Gamma_2 + u\Gamma_1') \sin H - B(\Gamma_1 - u\Gamma_2') \cos H]\tfrac{1}{2}B(\Gamma_2 \sin H + \Gamma_1 \cos H).\end{aligned}$$

Let $\tilde{D}^{\pm}_{B,a,\beta}$ be the self-adjoint operators associated with the forms $\tilde{b}^{\pm}_{B,a,\beta}$, respectively. By a straightforward computation, one can check the following claim.

Lemma 3.2. $M_B^* D^{\pm}_{B,a,\beta} M_B = \tilde{D}^{\pm}_{B,a,\beta}$.

The next step is to estimate $\tilde{D}^{\pm}_{B,a,\beta}$ by operators with separated variables. Denoting

$$\gamma_{+:} = \max_{[0,L]} |\gamma(\cdot)|$$

we define

$$\begin{aligned}N_B(a) := \max_{(s,u)\in[0,L]\times[-a,a]} &\big|B(\Gamma_2 + u\Gamma_1')(1 + u\gamma)^{-1} \cos H - B(\Gamma_1 - u\Gamma_2')(1 + u\gamma)^{-1} \sin H\\ &-B(1 + u\gamma)^{-2}(\Gamma_2 \cos H - \Gamma_1 \sin H) + B(1 + u\gamma)^{-2}(\Gamma_2 \sin H + \Gamma_1 \cos H)'u\big|\end{aligned}$$

and

$$M_B(a) := \max_{(s,u)\in[0,L]\times[-a,a]} \left|W_B(s, u) + \tfrac{1}{4}\gamma(s)^2\right|.$$

Let $\emptyset \neq I \subset \mathbb{R}$ be a compact interval. Then there is a positive K such that

$$N_B(a) + M_B(a) \leqslant Ka \qquad \text{for} \quad 0 < a < \frac{1}{2\gamma_+} \quad \text{and} \quad B \in I$$

where K is independent of a and B. For a fixed $0 < a < \frac{1}{2\gamma_+}$, we define

$$\begin{aligned}\hat{b}^{\pm}_{B,a,\beta}[f] = \int_0^L \int_{-a}^{a} \big\{ \big[(1 \mp a\gamma_+)^{-2} \pm \tfrac{1}{2} N_B(a) \big] |\partial_s f|^2 + |\partial_u f|^2 \\ + \big[-\tfrac{1}{4}\gamma(s)^2 \pm \tfrac{1}{2} N_B(a) \pm M_B(a) \big] |f|^2 \big\} \, \mathrm{d}u \, \mathrm{d}s \\ - \beta \int_0^L |f(s,0)|^2 \, \mathrm{d}s - \gamma_+ b_\pm \int_0^L (|f(s,a)|^2 + |f(s,-a)|^2) \, \mathrm{d}s\end{aligned}$$

for $f \in \tilde{Q}^{\pm}_{B,a}$, respectively. Since $|\mathrm{Im}(\bar{g}\partial_s g)| \leqslant \frac{1}{2}\big(|g|^2 + |\partial_s g|^2\big)$, we obtain

$$\tilde{b}^{+}_{B,a,\beta}[f] \leqslant \hat{b}^{+}_{B,a,\beta}[f] \qquad \text{for} \quad f \in \tilde{Q}^{+}_{B,a} \tag{3.3}$$

$$\hat{b}^{-}_{B,a,\beta}[f] \leqslant \tilde{b}^{-}_{B,a,\beta}[f] \qquad \text{for} \quad f \in \tilde{Q}^{-}_{B,a}. \tag{3.4}$$

Let $\hat{H}^{\pm}_{B,a,\beta}$ be the self-adjoint operators associated with the forms $\hat{b}^{\pm}_{B,a,\beta}$, respectively. Furthermore, let $T^{+}_{a,\beta}$ be the self-adjoint operator associated with the form

$$t^{+}_{a,\beta}[f] = \int_{-a}^{a} |f'(u)|^2 \, \mathrm{d}s - \beta |f(0)|^2 \qquad f \in H^1_0((-a,a))$$

and similarly, let $T^{-}_{a,\beta}$ be the self-adjoint operator associated with the form

$$t^{-}_{a,\beta}[f] = \int_{-a}^{a} |f'(u)|^2 \, \mathrm{d}s - \beta |f(0)|^2 - \gamma_+ (|f(a)|^2 + |f(-a)|^2) \qquad f \in H^1((-a,a)).$$

We define

$$U^{\pm}_{B,a} = -\left[(1 \mp a\gamma_+)^{-2} \pm \frac{1}{2} N_B(a) \right] \frac{\mathrm{d}^2}{\mathrm{d}s^2} - \frac{1}{4}\gamma(s)^2 \pm \frac{1}{2} N_B(a) \pm M_B(a)$$

in $L^2((0,L))$ with the domain P_B specified in the previous section. Then we have

$$\hat{H}^{\pm}_{B,a,\beta} = U^{\pm}_{B,a} \otimes 1 + 1 \otimes T^{\pm}_{a,\beta}. \tag{3.5}$$

Let $\mu^{\pm}_j(B,a)$ be the jth eigenvalue of $U^{\pm}_{B,a}$ counted with multiplicity. We shall prove the following estimate.

Proposition 3.3. *Let $j \in \mathbb{N}$. Then there exists $C(j) > 0$ such that*

$$|\mu^{+}_j(B,a) - \mu_j(B)| + |\mu^{-}_j(B,a) - \mu_j(B)| \leqslant C(j)a$$

holds for $B \in I$ and $0 < a < \frac{1}{2\gamma_+}$, where $C(j)$ is independent of B and a.

Proof. Since

$$\begin{aligned}U^{+}_{B,a} - \left[(1 - a\gamma_+)^{-2} + \frac{1}{2} N_B(a) \right] S_B \\ = \frac{1}{4} \left[\frac{a\gamma_+(2 - a\gamma_+)}{(1 - a\gamma_+)^2} + \frac{1}{2} N_B(a) \right] \gamma(s)^2 + \frac{1}{2} N_B(a) + M_B(a)\end{aligned}$$

and since $N_B(a) + M_B(a) \leqslant Ka$ for $0 < a < \frac{1}{2\gamma_+}$ and $B \in I$, we infer that there is a constant $C_1 > 0$ such that

$$\left\| U^{+}_{B,a} - \big[(1 - a\gamma_+)^{-2} + \tfrac{1}{2} N_B(a) \big] S_B \right\| \leqslant C_1 a$$

for $0 < a < \frac{1}{2\gamma_+}$ and $B \in I$. This together with the min–max principle implies that

$$\left|\mu_j^+(B,a) - \left[(1-a\gamma_+)^{-2} + \tfrac{1}{2}N_B(a)\right]\mu_j(B)\right| \leqslant C_1 a$$

for $0 < a < \frac{1}{2\gamma_+}$ and $B \in I$. Since $\mu_j(\cdot)$ is continuous, we claim that there exists a constant $C_2 > 0$ such that

$$|\mu_j^+(B,a) - \mu_j(B)| \leqslant C_2 a$$

for $0 < a < \frac{1}{2\gamma_+}$ and $B \in I$. In a similar way, we infer the existence of a constant $C_3 > 0$ such that

$$|\mu_j^-(B,a) - \mu_j(B)| \leqslant C_3 a$$

for $0 < a < \frac{1}{2\gamma_+}$ and $B \in I$. □

We also recall the following result from [EY].

Proposition 3.4. *(a) Suppose that $\beta a > \frac{8}{3}$. Then $T_{a,\beta}^+$ has only one negative eigenvalue, which we denote by $\zeta_{a,\beta}^+$. It satisfies the inequalities*

$$-\tfrac{1}{4}\beta^2 < \zeta_{a,\beta}^+ < -\tfrac{1}{4}\beta^2 + 2\beta^2 \exp\left(-\tfrac{1}{2}\beta a\right).$$

(b) Let $a\beta > 8$ and $\beta > \frac{8}{3}\gamma_+$. Then $T_{a,\beta}^-$ has a unique negative eigenvalue $\zeta_{a,\beta}^-$, and moreover, we have

$$-\tfrac{1}{4}\beta^2 - \tfrac{2205}{16}\beta^2 \exp\left(-\tfrac{1}{2}\beta a\right) < \zeta_{a,\beta}^- < -\tfrac{1}{4}\beta^2.$$

Proof of theorem 2.1. We put $a(\beta) = 6\beta^{-1}\ln\beta$. Let $\xi_{\beta,j}^\pm$ be the jth eigenvalue of $T_{a(\beta),\beta}^\pm$, by proposition 3.4 we have

$$\xi_{\beta,1}^\pm = \zeta_{a(\beta),\beta}^\pm \qquad \xi_{\beta,2}^\pm \geqslant 0.$$

From the decompositions (3.5) we infer that $\{\xi_{\beta,j}^\pm + \mu_k^\pm(B, a(\beta))\}_{j,k\in\mathbb{N}}$, properly ordered, is the sequence of the eigenvalues of $\hat{H}_{B,a(\beta),\beta}^\pm$ counted with multiplicity. Proposition 3.3 gives

$$\xi_{\beta,j}^\pm + \mu_k(B, a(\beta)) \geqslant \mu_1^\pm(B, a(\beta)) = \mu_1(B) + \mathcal{O}(\beta^{-1}\ln\beta) \tag{3.6}$$

for $B \in I$, $j \geqslant 2$, and $k \geqslant 1$, where the error term is uniform with respect to $B \in I$. For a fixed $j \in \mathbb{N}$, we put

$$\tau_{B,\beta,j}^\pm = \zeta_{a(\beta),\beta}^\pm + \mu_j^\pm(B, a(\beta)).$$

Combining propositions 3.3 and 3.4 we get

$$\tau_{B,\beta,j}^\pm = -\tfrac{1}{4}\beta^2 + \mu_j(B) + \mathcal{O}(\beta^{-1}\ln\beta) \qquad \text{as} \quad \beta \to \infty \tag{3.7}$$

where the error term is uniform with respect to $B \in I$. Let us fix now $n \in \mathbb{N}$. Combining (3.6) with (3.7) we infer that there exists $\beta(n, I) > 0$ such that the inequalities

$$\tau_{B,\beta,n}^+ < 0 \qquad \tau_{B,\beta,n}^+ < \xi_{\beta,j}^+ + \mu_k^+(B, a(\beta)) \qquad \tau_{B,\beta,n}^- < \xi_{\beta,j}^- + \mu_k^-(B, a(\beta))$$

hold for $B \in I$, $\beta \geqslant \beta(n, I)$, $j \geqslant 2$ and $k \geqslant 1$. Hence the jth eigenvalue of $\hat{H}_{B,a(\beta),\beta}^\pm$ counted with multiplicity is $\tau_{B,\beta,j}^\pm$ for $B \in I$, $j \leqslant n$ and $\beta \geqslant \beta(n, I)$. Let $B \in I$ and $\beta \geqslant \beta(n, I)$. We denote by $\kappa_j^\pm(B, \beta)$ the jth eigenvalue of $L_{B,a,\beta}^\pm$. Combining our basic estimate (3.1) with lemmas 3.1 and 3.2, relations (3.3) and (3.4), and the min–max principle, we arrive at the inequalities

$$\tau_{B,\beta,j}^- \leqslant \kappa_j^-(B, \beta) \qquad \text{and} \qquad \kappa_j^+(B, \beta) \leqslant \tau_{B,\beta,j}^+ \qquad \text{for} \quad 1 \leqslant j \leqslant n. \tag{3.8}$$

So we have $\kappa_n^+(B,\beta) < 0 < \inf \sigma_{\mathrm{ess}}(H_{B,\beta})$. Hence the min–max principle and (3.1) imply that $H_{B,\beta}$ has at least n eigenvalues in $(-\infty, \kappa_n^+(B,\beta)]$. Given $1 \leqslant j \leqslant n$, we denote by $\lambda_j(B,\beta)$ the jth eigenvalue of $H_{B,\beta}$. It satisfies

$$\kappa_j^-(B,\beta) \leqslant \lambda_j(B,\beta) \leqslant \kappa_j^+(B,\beta) \qquad \text{for} \quad 1 \leqslant j \leqslant n.$$

This together with (3.7) and (3.8) implies that

$$\lambda_j(B,\beta) = -\tfrac{1}{4}\beta^2 + \mu_j(B) + \mathcal{O}(\beta^{-1}\ln\beta) \qquad \text{as} \quad \beta \to \infty \quad \text{for} \quad 1 \leqslant j \leqslant n$$

where the error term is uniform with respect to $B \in I$. This completes the proof. □

Proof of corollary 2.2. By [RS, theorem XIII.89] no eigenvalue $\mu_n(\cdot)$ is constant on I. This together with theorem 2.1 yields the claim. □

4. Concluding remarks

The above corollary answers the question we posed in the introduction as a mathematical problem showing that a ring with a strong enough attractive δ-interaction does exhibit persistent currents. On the other hand, from the physical point of view it would be bold to identify a mere non-constantness of the eigenvalues with a genuine magnetic transport around the loop.

The problem is similar to other situations where an electron can be transported in a magnetic field due to the presence of a 'guiding' perturbation. A prime example is the edge currents [Ha, MS] which attracted a wave of mathematical interest recently in connection with the problem of stability of the transport with respect to perturbations. In the case of a single edge and a weak disorder, a part of the absolutely continuous spectrum survives [BP, FGW, MMP] but the fact itself gives no quantitative information about the transport. On the other hand, a system with more than one edge may have no continuous spectrum at all and still it has states in which electrons travel distances much larger than the corresponding cyclotron radius [FM].

In our case it is clear, for instance, that the loop geometry influences the transport substantially. If Γ is a circle, for example, then up to the $\mathcal{O}(\beta^{-1}\ln\beta)$ error the persistent-current plot will have the ideal saw-tooth shape as we can see from relation (1.1); one expects that the eigenfunctions will be 'spread' around the whole circle. In contrast, if the loop is rather 'wiggly' the one-dimensional comparison operator S_B contains an irregular effective potential coming from the rapidly varying curvature, which may cause—depending on the strength of such a 'disorder'—that the most part of the electron wavefunction will be concentrated in (the vicinity of) a part of the loop only. The same may happen if the loop curvature slowly changes but a disorder potential is added to the Hamiltonian.

To distinguish the situations with a *significant* transport, one needs clearly to understand better the sketched 'disordered' cases which do not fall into this category. We leave this problem to a future publication.

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References

[BP] de Bièvre S and Pulé J V 1999 Propagating edge states for a magnetic Hamiltonian *Math. Phys. Electron. J.* **5**

[BEKS] Brasche J F, Exner P, Kuperin Yu A and Seba P 1994 Schrödinger operators with singular interactions *J. Math. Anal. Appl.* **184** 112–39

[BT] Brasche J F and Teta A 1992 Spectral analysis and scattering theory for Schrödinger operators with an interaction supported by a regular curve *Ideas and Methods in Quantum and Statistical Physics* (Cambridge: Cambridge University Press) pp 197–211

[CGR] Cheng H F, Gefen Y, Riedel E K and Shih W H 1988 Persistent currents in small one-dimensional metal rings *Phys. Rev.* B **37** 6050–62

[CWB] Chadrasekhar V, Web R A, Brady M J, Ketchen M B, Gallagher W J and Kleinsasser A 1991 Magnetic response of a single, isolated gold loop *Phys. Rev. Lett.* **67** 3578–81

[EI] Exner P and Ichinose T 2001 Geometrically induced spectrum in curved leaky wires *J. Phys. A: Math. Gen.* A **34** 1439–50

[ES] Exner P and Seba P 1989 Bound states in curved quantum waveguides *J. Math. Phys.* **30** 2574–80

[EY] Exner P and Yoshitomi K 2002 Asymptotics of eigenvalues of the Schrödinger operator with a strong δ-interaction on a loop *J. Geom. Phys.* **41** 344–58 (math-ph/0103029 and mp-arc 01-108)

[FGW] Fröhlich J, Graf G M and Walcher J 2000 On the extended nature of edge states of quantum Hall Hamiltonians *Ann. H Poincaré* **1** 405–42

[FM] Ferrari Ch and Macris N 2000 Intermixture of extended edge and localized bulk energy levels in macroscopic Hall systems *Preprint* math-ph/0011013

[Ha] Halperin B I 1982 Quantized Hall conductance, current carrying edge states, and the existence of extended states in two-dimensional disordered potential *Phys. Rev.* B **25** 2185–90

[MS] MacDonald A H and Středa P 1984 Quantized Hall effect and edge currents *Phys. Rev.* B **29** 1616–9

[MMP] Macris N, Martin Ph A and Pulé J V 1999 On edge states in semi-infinite quantum Hall systems *J. Phys. A: Math. Gen.* **32** 1985–96

[RS] Reed M and Simon B 1978 *Methods of Modern Mathematical Physics: IV. Analysis of Operators* (New York: Academic)